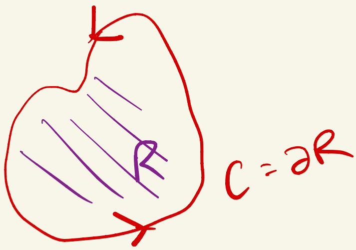
9-3:

Recall: Big Theme

$$\iint_R \text{"derivative } F\text{"} \cdot dA = \int_{\partial R} \text{"} F \cdot d\vec{r} \text{"}$$

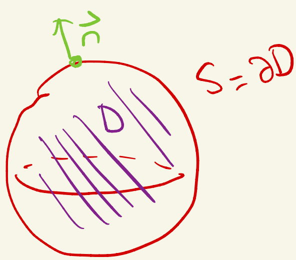
R ↑ surface

∂R ↑ boundary ~~of~~ surface



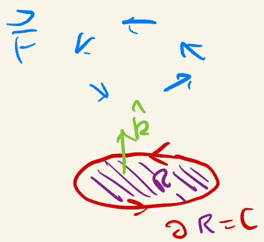
and

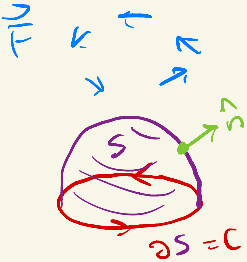
$$\iiint_D \text{"derivative } F\text{"} \cdot dV = \iint_S \text{"} F \cdot \vec{n} \text{"} \, d\sigma$$



Extending Green's theorems to 3D

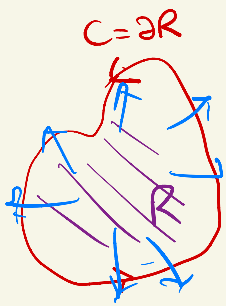
- Stokes's theorem extends Green's curl-circulation theorem to <u>any</u> surface R with boundary $\partial R = C$:

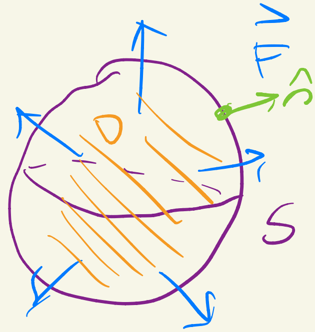




Green's (tangential)	Stokes' Theorem
$\iint_R (\nabla \times \vec{F}) \cdot \hat{k}\, dA = \oint_{\partial R} \vec{F} \cdot d\vec{r}$	$\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, d\sigma = \oint_{\partial S} \vec{F} \cdot d\vec{r}$

Green's Thm (normal)	Divergence Theorem
$\iint_R (\nabla \cdot \vec{F})\, dx\, dy = \oint_{\partial R} \vec{F} \cdot \vec{n}\, ds$	$\iiint_D \nabla \cdot \vec{F}\, dV = \iint_S \vec{F} \cdot \hat{n}\, d\sigma$
Total divergence inside = outward flux boundary	Total divergence inside = outward flux boundary





16.7: Stokes and Divergence Theorems

Reminder: pay attention to "type" of object.

- $f(x,y,z)$ ← scalar function
- $\vec{F}(x,y,z) = \langle M, N, P \rangle$ ← vector field

  Notation: $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$

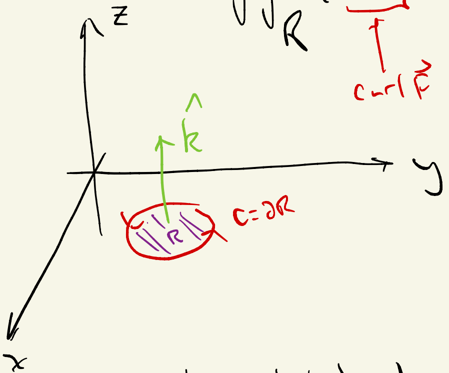
- Gradient:
  $$\nabla f = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$ ← vector field

- Divergence:
  $$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$ ← scalar function

- Curl:
  $$\operatorname{curl} \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$ ← vector field
  $$= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial x} \right) \hat{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z} \right) \hat{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \hat{k}$$

Recall: Green's thm (curl-circulation form)

$$\iint_R \underbrace{(\nabla\times\vec{F})}_{\text{curl }\vec{F}}\cdot\hat{k}\,dA = \oint_C \vec{F}\cdot d\vec{r}$$

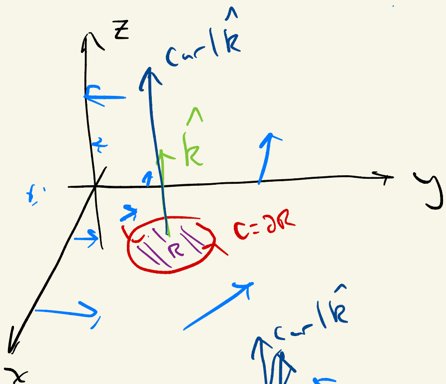


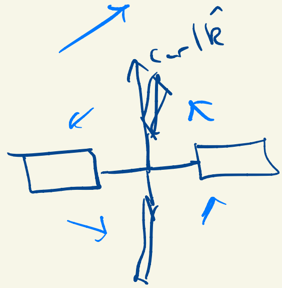
Good example: whirlpool

$\vec{F} = -y\hat{i} + x\hat{j}$

$\text{curl }\vec{F} = 2\hat{k}$

$$\oint_{\text{unit circle ccw}} \vec{F}\cdot d\vec{r} = \underbrace{2}_{\text{curl }\vec{F}}\underbrace{\pi}_{\text{area unit circle}}$$





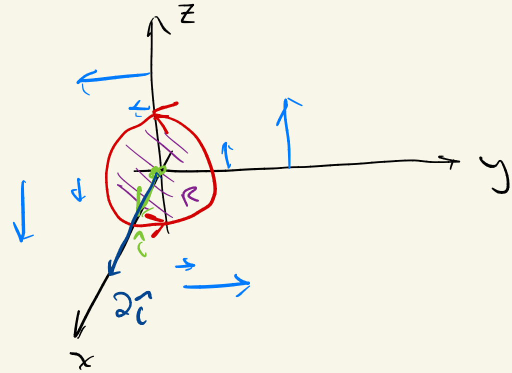
What about other whirlpools?

$\vec{F} = -z\hat{j} + y\hat{k}$

$$\text{curl }\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & -z & y \end{vmatrix}$$

$$= (1+1)\hat{i}$$

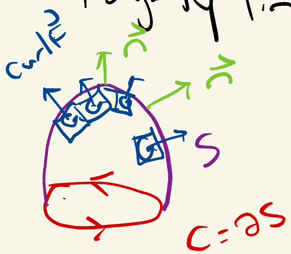
$$= 2\hat{i}$$



Idea: curl $\vec{F}$ detects planar rotation of $\vec{F}$ and places it in normal vector to plane of rotation

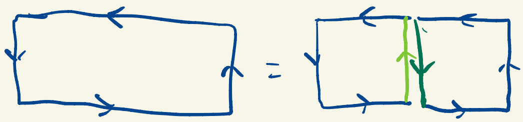
Now: take any surface S with boundary $\partial S = C$.

can break into tiny pieces which look roughly linear (like little planes!)



orientation of S is determined by right hand rule on oriented curve C

Like Green's thm:



"3D Green's Thm"

$\vec{n}$ determines orientation "inside" vs. "outside"

Stokes' Theorem

Let S be a piecewise smooth oriented surface with a piecewise smooth boundary $C = \partial S$.
If $\vec{F}$ is a vector field with continuous 1st derivatives, then

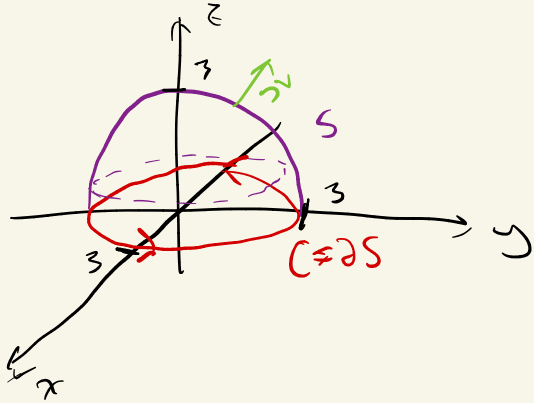
$$\oint_{\substack{\partial S \\ C = \partial S}} \vec{F} \cdot d\vec{r} = \iint_S \underbrace{(\nabla \times \vec{F})}_{\text{curl } \vec{F}} \cdot \vec{n} \, d\sigma$$

Ex:

Compute $\iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, d\sigma$ where

$S =$ half sphere
$x^2+y^2+z^2 = 9$
$z \geq 0$

$\vec{F} = y\hat{i} - x\hat{j}$



Soln:

Option 1: Do the surface integral!
Option 2: Stokes:

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, d\sigma = \oint_C \vec{F} \cdot d\vec{r}$$

calculate this!

$\vec{r}(t) = 3\cos t \, \hat{i} + 3\sin t \, \hat{j}$
$0 \leq t \leq 2\pi$

$F = y\hat{i} - x\hat{j}$

$\vec{r}\,'(t) = -3\sin t \, \hat{i} + 3\cos t \, \hat{j}$

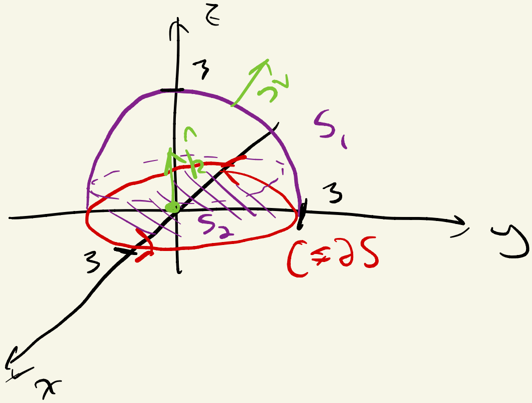
$$\vec{F} \cdot \vec{r}\,'(t) = (3\sin t \, \hat{i} - 3\cos t \, \hat{j}) \cdot (-3\sin t \, \hat{i} + 3\cos t \, \hat{j})$$
$$= -9\sin^2 t - 9\cos^2 t$$
$$= -9$$

$$\Rightarrow \iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, d\sigma = \oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} -9 \, dt$$
$$= -18\pi$$

Surprise Option 3:

S_1 = half sphere

S_2 = disc of radius 3 in xy



$$\iint_{S_1} (\nabla \times \vec{F}) \cdot \vec{n} \, d\sigma = \oint_C \vec{F} \cdot d\vec{r}$$

stokes

$$= \iint_{S_2} (\nabla \times \vec{F}) \cdot \vec{n} \, d\sigma$$

$$= \iint_{S_2} (\nabla \times \vec{F}) \cdot \hat{k} \, dA$$

since S_2 in xy plane

$$\nabla \times \vec{F} = \left(\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}(y) \right) \hat{k}$$

$$= -2\hat{k}$$

For this problem, $\nabla \times \vec{F}$ only has a $\hat{k}$ component. However, if $\nabla \times \vec{F}$ had more than a $\hat{k}$ component, since we care about $(\nabla \times \vec{F}) \cdot \hat{k}$, only need $\hat{k}$ component because everything else will die.

$$\Rightarrow \iint_{S_2} (\nabla \times \vec{F}) \cdot \hat{k} \, dA = \iint_{\bigcirc_3} (-2\hat{k}) \cdot \hat{k} \, dA$$

$$= -2 \iint_{\bigcirc_3} dA$$

$$= -2(3^2\pi) = \boxed{-18\pi}$$

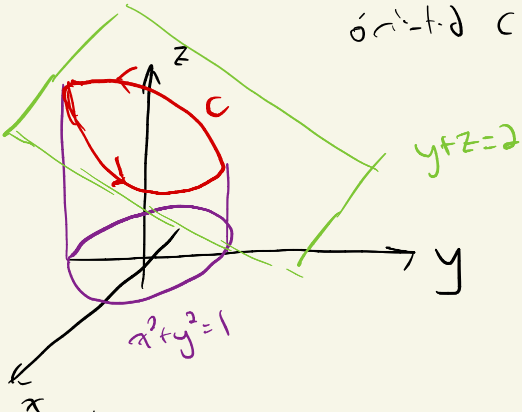
Ex:

Evaluate $\oint_C \vec{F}\cdot d\vec{r}$, $\vec{F} = -y^2\hat{i} + x\hat{j} + z^2\hat{k}$

C = intersection of plane $y+z=2$

cylinder $x^2+y^2=1$

oriented CCW from above



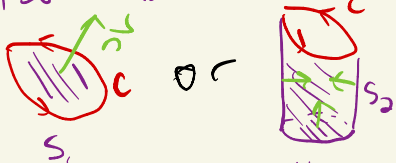
Soln:

Option 1: Do the line integral (parametrizing won't be that easy.)
Not too hard though

Option 2: Stokes theorem:

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\vec{n}\, d\sigma$$

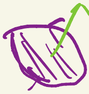
two choices



Depending on the vector field, more complicated surfaces can have easier integrals

$$\nabla\times\vec{F} = \left(\cancelto{0}{\frac{\partial}{\partial y}(z^2) - \frac{\partial}{\partial z}(x)}\right)\hat{i} - \left(\cancelto{0}{\frac{\partial}{\partial x}(z^2) - \frac{\partial}{\partial z}(-y^2)}\right)\hat{j} + \left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y^2)\right)\hat{k}$$

$$= (1+2y)\hat{k}$$

S_1  $\vec{n}$

well, level surface

$g(x,y,z) = y+z-2=0$

$$\Rightarrow \vec{n} = \frac{\nabla g}{|\nabla g|} = \frac{1}{\sqrt{2}}\hat{j} + \frac{1}{\sqrt{2}}\hat{k}$$

Explicit surface:

$$\iint_{S_1} (\nabla\times\vec{F})\cdot\vec{n}\,d\sigma = \iint_{\substack{\text{unit}\\ \text{disc in } xy}} (1+2y)\hat{k}\cdot\left(\frac{1}{\sqrt{2}}\hat{j}+\frac{1}{\sqrt{2}}\hat{k}\right)\sqrt{f_x^2+f_y^2+1}\,dA$$

$f(x,y) = z = -y+2$

$$= \iint_{\substack{\text{unit}\\ \text{disc}}} \cancel{\frac{1}{\sqrt{2}}}(1+2y)\cdot\cancel{\sqrt{0^2+(-1)^2+1}}\,dA$$

$$= \int_0^{2\pi}\int_0^1 (1+2r\sin\theta)\,r\,dr\,d\theta$$

$$= \pi$$

level surface for cylinder: $x^2+y^2=1 \Rightarrow G(x,y) = x^2+y^2$

$\vec{n} = \frac{\nabla G}{|\nabla G|}$

$\vec{n} \perp \hat{k}$

$\Rightarrow \vec{n}\cdot\hat{k} = 0$ on sides

S_2 

$\vec{n} = \hat{k}$

$$\iint_{S_2} (\nabla\times\vec{F})\cdot\vec{n}\,d\sigma = \iint_{\text{bottom}} (1+2y)\hat{k}\cdot\vec{n}\,d\sigma + \iint_{\text{sides}} (1+2y)\hat{k}\cdot\vec{n}\,d\sigma$$

$$= \iint_{\substack{\text{unit}\\ \text{disc}}} (1+2y)\hat{k}\cdot\hat{k}\,dA + 0$$

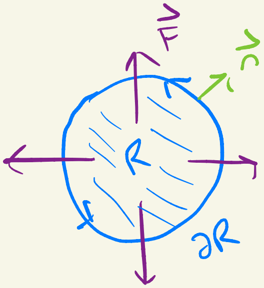
$$= \iint_{\substack{\text{unit}\\ \text{disc}}} 1+2y\;dx\,dy$$

$$= \pi$$

16.8 Divergence Theorem

Recall: Green's thm (flux-divergence form)

$$\oint_{\partial R} \vec{F}\cdot\vec{n}\,ds = \iint_R \underbrace{\nabla\cdot\vec{F}}_{\operatorname{div}\vec{F}}\,dA$$



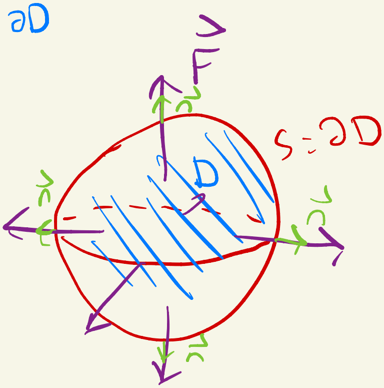
"flux on boundary detects amount of source/sink inside"

Divergence theorem says this idea should work in 3D as well!

Divergence Theorem:

Let $\vec{F}$ a vector field with continuous first derivatives on S a piecewise smooth closed surface enclosing D. The outward flux of $\vec{F}$ across S is

$$\underset{\substack{S \\ \partial D}}{\oiint} \vec{F}\cdot\vec{n}\,d\sigma = \iiint_D \underbrace{\nabla\cdot\vec{F}}_{\operatorname{div}\vec{F}}\,dV$$

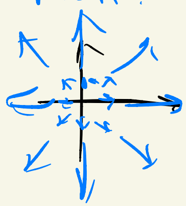


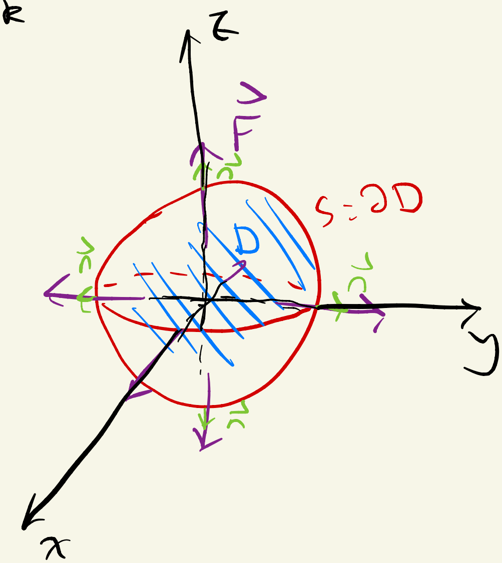
Ex!

Find the flux of $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$ through sphere $x^2+y^2+z^2=a^2$

Soln:

Recall: classic "Source" example $\vec{F} = x\hat{i} + y\hat{j}$





Option 1: Do the surface integral

Implicit: $g(x,y,z) = x^2+y^2+z^2-a^2 = 0$

$$\Rightarrow \vec{n} = \frac{\nabla g}{|\nabla g|} = \frac{2(x\hat{i}+y\hat{j}+z\hat{k})}{\sqrt{4(x^2+y^2+z^2)}} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{a}$$

same direction as $\vec{F}$

$$\Rightarrow \vec{F}\cdot\vec{n} = |\vec{F}|\cdot|\vec{n}| = |\vec{F}|$$

unit vector in $\vec{F}$ direction

$|\vec{n}| = 1$

$= \sqrt{x^2+y^2+z^2}$

$= a$

$$\oiint_S \vec{F}\cdot\vec{n}\,d\sigma = \oiint_S a\,d\sigma$$

$$= a\oiint_S d\sigma$$

$= a \cdot$ surface area

$= 4\pi a^2 \cdot a$

$= 4\pi a^3$

Option 2: Divergence Thm

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$$

$$\oiint_S \vec{F} \cdot \vec{n}\, d\sigma = \iiint_D 3\, dV = 3 \iiint_D dV$$

$$= 3\, \mathrm{Vol}(\text{sphere})$$

$$= 3\left(\frac{4}{3}\pi a^3\right)$$

$$= 4\pi a^3$$

$\blacksquare$

In $\mathbb{R}^3$:

$$\text{scalar field } f \xrightarrow{\nabla f \text{ gradient}} \text{vector field } \vec{F} \xrightarrow{\nabla \times \vec{F} \text{ curl}} \text{vector field } \vec{G} \xrightarrow{\nabla \cdot \vec{F} \text{ divergence}} \text{scalar field } g$$

$$\nabla \times (\nabla f) = 0$$

"curl of gradient is 0"

(conservative!)

$$\nabla \cdot (\nabla \times \vec{F}) = 0$$

"divergence of the curl is 0"

<u>Def:</u>

Irrotational: $\nabla \times \vec{F} = 0$ ← component test

(curless)

(aka $\vec{F}$ conservative)

Incompressible: $\nabla \cdot \vec{F} = 0$

(divergentless)

<u>Consequence:</u>

- Gradients are irrotational
- curls are incompressible